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Non-standard covariant deformed oscillator algebras

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Abstract. It is shown that deformed oscillator algebras with a non-standard behaviour, similar to that observed for the quon algebra, arise quite naturally when imposing covariance properties under the transformations of some quantum group. By non-standard behaviour, we mean that there may not be enough rules available to order creation operators and that some Fock states with positive squared norm for generic q-values disappear for $q \to 1$. Two examples are given for the quantum groups $O_q(3,\mathbb{R})$ and $SU_q(2) \times SU_q(2) \simeq O_q(4,\mathbb{R})$, respectively.

It has been known for decades that Bose and Fermi statistics are not the only ones allowed within the framework of quantum mechanics [1]. With the advent of quantum groups and q-algebras (see e.g. [2] and references quoted therein), q-deformations of the canonical commutation or anticommutation relations have recently become a subject of intense study [3–12]. Deformed commutators are used, for instance, in the quon statistics [13] which has aroused considerable interest during the last few years because it provides a smooth interpolation between Bose and Fermi statistics.

The quon algebra, depending on a single real parameter q, has a representation in a Fock space with non-negative scalar products as long as q lies in the range $-1 \leqslant q \leqslant +1$ [13]. For $-1 < q < +1$, all irreducible representations of the symmetric group have a positive squared norm, whereas for $q = +1$ (respectively $q = -1$), only the symmetric (respectively antisymmetric) representation survives. Hence, there is no one-to-one correspondence between Fock states corresponding to generic q values and those obtained for $q = +1$ or -1.

Another remarkable (and related) feature of the quon algebra is that no commutation relation is (or can be) imposed on two creation or two annihilation operators. This contrasts with some other constructions of deformed commutators wherein relations between all pairs of operators are required from the very beginning [6].

The purpose of the present paper is to show that deformed oscillator algebras sharing some features with the quon algebra arise quite naturally when imposing covariance properties under the transformations of some quantum group. It should be stressed that such features are absent in those covariant algebras that have been considered so far, namely q-bosonic algebras covariant under $SU_q(n)$ [5], $SU_q(n) \times SU_q(m)$ [9, 10] or $O_q(N,\mathbb{R})$ [11, 12] and q-fermionic algebras covariant under $SU_q(n)$ [5] or $USp_q(2n)$ [11, 12]. In all these cases, definite rules are indeed available for transposing any two operators and q-bosonic (respectively q-fermionic) Fock states can be put into one-to-one correspondence with their bosonic (respectively fermionic) counterparts.

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The first example we shall deal with is an $O_q(3,\mathbb{R})$-covariant algebra spanned by three pairs of creation and annihilation operators $B_m^\dagger$, $B_m = (B_m^\dagger)^\dagger$, $m = +1, 0, -1$ and the unit operator. As in [11], the operators $B_m^\dagger$ and $\tilde{B}_m = (-1)^m q^{m/2} B_{-m}$, where $q \in \mathbb{R}^+$, are the components of vector operators with respect to the q-algebra $so_q(3)$, but instead of fulfilling the coupled q-commutation relations (7) given in [11], they are assumed to satisfy the relations

$$\{B^\dagger, B^\dagger\}^2 = \{\tilde{B}, \tilde{B}\}^2 = 0 \tag{1a}$$

$$\{\tilde{B}, B^\dagger\}^2_{q^{-4}} = \{\tilde{B}, B^\dagger\}^1 = 0 \qquad \{\tilde{B}, B^\dagger\}^0_{q^2} = -\sqrt{[3]_q}\, I. \tag{1b}$$

Here $[n]_q = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$ denotes a q-number and $\{T^\lambda, U^{\lambda'}\}^\Lambda_{Mq^\alpha}$ is an $so_q(3)$ coupled q-anticommutator of the irreducible tensors T^λ and $U^{\lambda'}$ defined by

$$\{T^\lambda, U^{\lambda'}\}^\Lambda_{Mq^\alpha} = [T^\lambda \times U^{\lambda'}]^\Lambda_M + (-1)^{\lambda+\lambda'-\Lambda} q^{\alpha/2} [U^{\lambda'} \times T^\lambda]^\Lambda_M \tag{2}$$

with

$$[T^\lambda \times U^{\lambda'}]^\Lambda_M = \sum_{\mu\mu'} \langle \lambda\mu, \lambda'\mu' \mid \Lambda M\rangle_q T^\lambda_\mu U^{\lambda'}_{\mu'} \tag{3}$$

and $\langle\, ,\, |\,\rangle_q$ an $su_q(2) \simeq so_q(3)$ Wigner coefficient. Note that in (1), $\lambda = \lambda' = 1$ and, for simplicity's sake, label M has been skipped.

By using the numerical values of the Wigner coefficients [14] and ordinary q-anticommutators defined by $\{X, Y\}_{q^\alpha} = XY + q^{\alpha/2} YX$, relations (1) can be written in explicit form as

$$\{B_1^\dagger, B_1^\dagger\} = \{B_{-1}^\dagger, B_{-1}^\dagger\} = 0 \tag{4a}$$

$$\{B_1^\dagger, B_0^\dagger\}_{q^2} = \{B_0^\dagger, B_{-1}^\dagger\}_{q^2} = 0 \tag{4b}$$

$$\{B_1^\dagger, B_{-1}^\dagger\}_{q^4} = -(q^{3/2} + q^{1/2})(B_0^\dagger)^2 \tag{4c}$$

$$\{B_{-1}, B_1^\dagger\}_{q^{-4}} = \{B_0, B_1^\dagger\}_{q^{-2}} = 0 \tag{4d}$$

$$\{B_{-1}, B_0^\dagger\}_{q^{-2}} = (q^{-5/2} - q^{-1/2}) B_1^\dagger B_0 \tag{4e}$$

$$\{B_1, B_1^\dagger\} = I \qquad \{B_0, B_0^\dagger\}_{q^{-2}} = I + (q^{-2} - 1) B_1^\dagger B_1 \tag{4f}$$

$$\{B_{-1}, B_{-1}^\dagger\} = I + (q^{-2} - 1)[(1 - q^{-1}) B_1^\dagger B_1 + B_0^\dagger B_0] \tag{4g}$$

together with the Hermitian conjugates of (4a)–(4e).

One procedure for constructing the algebra defined in (1) or (4) is based upon the fundamental R-matrix of the quantum group $O_q(3,\mathbb{R})$ [2]. The corresponding braid matrix $\hat{R}$ has three eigenspaces characterized by 'angular momentum' $L = 2$, 1 or 0 corresponding to the eigenvalues q, $-q^{-1}$ and q^{-2}, respectively. As shown in [12], an associative deformed oscillator algebra can be constructed by imposing that the relations among creation (or annihilation) operators be determined by a projector onto a braid-matrix eigenspace: for the q-bosonic operators considered in [12] (or [11]), it is the eigenspace characterized by $L = 1$, whereas, here, we use the eigenspace specified by $L = 2$ (see (1a)). Once the projector has been chosen, it turns out that the associativity condition entirely fixes (up to a change of q into q^{-1}) the commutation properties of the annihilation operators with the creation operators. The resulting set of relations remains invariant under the coaction of $O_q(3,\mathbb{R})$, thus showing that the algebra is $O_q(3,\mathbb{R})$-covariant.

Alternatively, as in [11], the defining relations (1) of the algebra can be derived by using the q-algebra $so_q(3)$. First, one checks that relations (1*a*) or (4*a*)–(4*c*) among the creation operators satisfy associativity by proving that the two braid transposition schemes starting from $(B_{-1}^\dagger B_0^\dagger)B_1^\dagger$ and $B_{-1}^\dagger(B_0^\dagger B_1^\dagger)$, respectively, lead to the same result. Then, in the remaining relations, the q-dependence is fixed by imposing associativity through the use of Racah's sum rule generalized to $su_q(2) \simeq so_q(3)$ [14]. In this $so_q(3)$ approach, irreducible tensor coupling automatically ensures covariance.

It is clear that equation (4) provides us with enough relations to normally order any product of creation and annihilation operators, i.e. to put the operators in a standard lexicographical ordering so that creation operators go to the left, destruction to the right, e.g. $(B_1^\dagger)^{n_1}(B_0^\dagger)^{n_0}(B_{-1}^\dagger)^{n_{-1}}(B_{-1})^{n'_{-1}}(B_0)^{n'_0}(B_1)^{n'_1}$. From this viewpoint, the present covariant deformed oscillator algebra does not differ from those previously studied [5, 8–12].

A first discrepancy appears when considering the $q \to 1$ limit of (4). We indeed obtain the defining relations of fermion operators

$$\{B_m^\dagger, B_{m'}^\dagger\} = \{B_m, B_{m'}\} = 0 \qquad \{B_m, B_{m'}^\dagger\} = \delta_{m,m'}I \tag{5}$$

except for the relations $\{B_0^\dagger, B_0^\dagger\} = \{B_0, B_0\} = 0$, which are missing. As a matter of fact, to obtain equation (5) for any values of m and m' in the $q \to 1$ limit, the defining relations of the algebra should contain, in addition to (1), the equation

$$\{B^\dagger, B^\dagger\}^0 = 0 \qquad \text{or} \qquad \{B_1^\dagger, B_{-1}^\dagger\}_{q^{-2}} = q^{-1/2}(B_0^\dagger)^2 \tag{6}$$

and its Hermitian conjugate. Combining (6) with (4*c*) would lead to the two equivalent relations

$$\{B_1^\dagger, B_{-1}^\dagger\} = 0 \qquad (B_0^\dagger)^2 = (q^{1/2} - q^{-1/2})B_1^\dagger B_{-1}^\dagger. \tag{7}$$

By working out the two braid transposition schemes for $B_1 B_{-1}^\dagger B_1^\dagger$, for instance, it is easy to check that, except in the $q \to 1$ limit, associativity is not preserved by the set of relations (4*a*), (4*b*), (4*d*)–(4*g*) and (7). Hence, equation (6) cannot be added to (1) for $q \neq 1$. This also follows from the fact that for $q \neq 1$, the braid-matrix eigenspaces with $L = 2$ and $L = 0$ correspond to distinct eigenvalues.

A second discrepancy with respect to standard covariant deformed algebras appears when considering the Fock space that can be constructed by acting with the operators $B_m^\dagger$ on the vacuum $|0\rangle$, i.e. the state annihilated by B_m, $m = +1, 0, -1$ such that $\langle 0|0\rangle = 1$. From (4*a*), it follows that we may restrict ourselves to the monomial states

$$|n_1 n_0 n_{-1}) = (B_1^\dagger)^{n_1}(B_0^\dagger)^{n_0}(B_{-1}^\dagger)^{n_{-1}}|0\rangle \qquad n_1, n_{-1} = 0, 1 \qquad n_0 = 0, 1, 2, \ldots. \tag{8}$$

It is a simple matter to prove that

$$(n'_1 n'_0 n'_{-1}|n_1 n_0 n_{-1}) = \delta_{n'_1,n_1}\delta_{n'_0,n_0}\delta_{n'_{-1},n_{-1}}\{n_0\}_p! \tag{9}$$

where $p = -q^{-1}$, $\{n\}_p = (1-p^n)/(1-p)$, $\{n\}_p! = \{n\}_p\{n-1\}_p \ldots \{1\}_p$ for $n \in \mathbb{N}^+$ and $\{0\}_p! = 1$. As long as $q > 1$, the overlap matrix of states (8) is therefore positive definite and the states

$$\begin{aligned} &|n_1 n_0 n_{-1}\rangle = (\{n_0\}_p!)^{-1/2}(B_1^\dagger)^{n_1}(B_0^\dagger)^{n_0}(B_{-1}^\dagger)^{n_{-1}}|0\rangle \\ &n_1, n_{-1} = 0, 1 \qquad n_0 = 0, 1, 2, \ldots \end{aligned} \tag{10}$$

form a Fock-space orthonormal basis. While the operators $B_1^\dagger$, $B_{-1}^\dagger$ are Fermi-like, $B_0^\dagger$ is instead Bose-like†. In the $q \to 1$ or $p \to -1$ limit, however, since $\{2\}_p \to 0$, only the states $|n_1 n_0 n_{-1}\rangle$ with $n_0 = 0$ or 1 survive so that $B_0^\dagger$ becomes a fermion operator.

The second example we shall deal with is an $SU_q(2) \times SU_q(2)$ (or equivalently $O_q(4, \mathbb{R})$)-covariant algebra, spanned by the unit operator and four pairs of creation and annihilation operators $A_{is}^\dagger$, $A_{is} = (A_{is}^\dagger)^\dagger$, i, $s = 1, 2$, which will also be denoted by $B_\mu^\dagger$, $B_\mu = (B_\mu^\dagger)^\dagger$, $\mu = 1, 2, 3, 4$:

$$B_1^\dagger = A_{11}^\dagger \qquad B_2^\dagger = A_{12}^\dagger \qquad B_3^\dagger = A_{21}^\dagger \qquad B_4^\dagger = A_{22}^\dagger. \tag{11}$$

The former notation is adapted to $SU_q(2) \times SU_q(2)$ while the latter corresponds to $O_q(4, \mathbb{R})$ after performing the substitutions $q \to q^2$, $B_1^\dagger \to B_1^\dagger$, $B_2^\dagger \to iB_2^\dagger$, $B_3^\dagger \to iB_3^\dagger$, $B_4^\dagger \to B_4^\dagger$. As in [9] and [10], the operators $A_{is}^\dagger$ and $\tilde{A}_{is} = (-1)^{i+s} q^{(3-i-s)/2} A_{i's'}$‡ where $i' = 3 - i$, $s' = 3 - s$ and $q \in \mathbb{R}^+$ are components of double spinors with respect to the q-algebra $su_q(2) + su_q(2)$, but instead of fulfilling equation (22) of [9] (for $n = m = 2$), they are assumed to satisfy the relations

$$\{A^\dagger, A^\dagger\}^{1,1} = \{\tilde{A}, \tilde{A}\}^{1,1} = 0 \tag{12a}$$

$$\{\tilde{A}, A^\dagger\}^{1,1}_{q^{-2}} = \{\tilde{A}, A^\dagger\}^{1,0} = \{\tilde{A}, A^\dagger\}^{0,1} = 0 \qquad \{\tilde{A}, A^\dagger\}^{0,0}_{q^2} = [2]_q I \tag{12b}$$

where

$$\{T^{\lambda_1\lambda_2}, U^{\lambda'_1\lambda'_2}\}^{\Lambda_1\Lambda_2}_{M_1M_2q^\alpha} = [T^{\lambda_1\lambda_2} \times U^{\lambda'_1\lambda'_2}]^{\Lambda_1\Lambda_2}_{M_1M_2} + (-1)^{\lambda_1+\lambda_2+\lambda'_1+\lambda'_2-\Lambda_1-\Lambda_2} q^{\alpha/2} [U^{\lambda'_1\lambda'_2} \times T^{\lambda_1\lambda_2}]^{\Lambda_1\Lambda_2}_{M_1M_2} \tag{13}$$

and $[T^{\lambda_1\lambda_2} \times U^{\lambda'_1\lambda'_2}]^{\Lambda_1\Lambda_2}_{M_1M_2}$ is given by an equation similar to (3) but containing two $su_q(2) \simeq so_q(3)$ Wigner coefficients instead of one.

Taking (11) into account, relations (12) can be written in explicit form as

$$\{B_1^\dagger, B_1^\dagger\} = \{B_2^\dagger, B_2^\dagger\} = \{B_3^\dagger, B_3^\dagger\} = \{B_4^\dagger, B_4^\dagger\} = 0 \tag{14a}$$

$$\{B_1^\dagger, B_2^\dagger\}_q = \{B_1^\dagger, B_3^\dagger\}_q = \{B_2^\dagger, B_4^\dagger\}_q = \{B_3^\dagger, B_4^\dagger\}_q = 0 \tag{14b}$$

$$\{B_1^\dagger, B_4^\dagger\}_{q^2} = -q^{1/2}(B_2^\dagger B_3^\dagger + B_3^\dagger B_2^\dagger) \tag{14c}$$

$$\{B_1, B_2^\dagger\}_{q^{-1}} = \{B_1, B_3^\dagger\}_{q^{-1}} = \{B_1, B_4^\dagger\}_{q^{-2}} = \{B_2, B_3^\dagger\}_{q^{-2}} = 0 \tag{14d}$$

$$\{B_2, B_4^\dagger\}_{q^{-1}} = (q^{-3/2} - q^{-1/2}) B_3^\dagger B_1 \tag{14e}$$

$$\{B_3, B_4^\dagger\}_{q^{-1}} = (q^{-3/2} - q^{-1/2}) B_2^\dagger B_1 \tag{14f}$$

$$\{B_1, B_1^\dagger\} = I \qquad \{B_2, B_2^\dagger\} = \{B_3, B_3^\dagger\} = I + (q^{-1} - 1) B_1^\dagger B_1 \tag{14g}$$

$$\{B_4, B_4^\dagger\} = I + (q^{-1} - 1)[(1 - q^{-1}) B_1^\dagger B_1 + B_2^\dagger B_2 + B_3^\dagger B_3^\dagger] \tag{14h}$$

together with the Hermitian conjugates of (14a)–(14f).

† Note also that the mixture of Bose-like and Fermi-like operators obtained for $q > 1$, although already observed before in connection with some non-standard R-matrices [8], is new in the standard R-matrix context (disregarding the case of quantum supergroups).

‡ Here, we use the notation of [10] for $\tilde{A}_{is}$, which slightly differs from that of [9].

The algebra defined in (12), or (14), can be constructed by using either the q-algebraic technique or one of the two R-matrix formulations that were employed for q-bosonic operators in [9] and [10], respectively. Some comments about associativity, analogous to those for the algebra defined in (1) or (4), could be made here. We shall instead go directly to the Fock-state construction where some new features make their appearance.

It is clear that equations (14a)–(14c) do not provide enough relations to put any product of creation operators in lexicographical order, no rule being available to reorder $B_3^\dagger B_2^\dagger$. It can be shown that any monomial state

$$|\mu_1\mu_2\dots\mu_n\rangle = B_{\mu_1}^\dagger B_{\mu_2}^\dagger \dots B_{\mu_n}^\dagger|0\rangle \qquad \mu_1, \mu_2, \dots, \mu_n = 1, \dots, 4 \tag{15}$$

with $n \geqslant 3$ can be rewritten in terms of the following states:

$$\begin{array}{llllll} |1(23)^\nu\rangle & |1(23)^{\nu-1}24\rangle & |1(32)^\nu\rangle & |1(32)^{\nu-1}34\rangle & |(23)^\nu 2\rangle & \\ |(23)^\nu 4\rangle & |(32)^\nu 3\rangle & |(32)^\nu 4\rangle & \text{if } n = 2\nu+1 & & \end{array} \tag{16a}$$

$$\begin{array}{llllll} |1(23)^{\nu-1}2\rangle & |1(23)^{\nu-1}4\rangle & |1(32)^{\nu-1}3\rangle & |1(32)^{\nu-1}4\rangle & |(23)^\nu\rangle & \\ |(23)^{\nu-1}24\rangle & |(32)^\nu\rangle & |(32)^{\nu-1}34\rangle & \text{if } n = 2\nu & & \end{array} \tag{16b}$$

the one- and two-particle states being given by $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ and $|12\rangle$, $|13\rangle$, $|14\rangle$, $|23\rangle$, $|24\rangle$, $|32\rangle$, $|34\rangle$, respectively. In (16), $(23)^\nu$, for instance, stands for the product $(B_2^\dagger B_3^\dagger)^\nu$.

The overlap matrix of the monomial states can be recursively determined. For (16a) and (16b), the results are

$$\begin{pmatrix} M_{2\nu}(q) & 0 \\ 0 & M_{2\nu+1}(q) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} M_{2\nu-1}(q) & 0 \\ 0 & M_{2\nu}(q) \end{pmatrix} \tag{17}$$

respectively, where, for $\nu = 1, 2, \dots,$

$$M_{2\nu}(q) = [[\nu-1]]_q! \begin{pmatrix} 1 & 0 & -q^{-\nu} & 0 \\ 0 & 1 & 0 & 0 \\ -q^{-\nu} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18a}$$

$$M_{2\nu+1}(q) = [[\nu-1]]_q! \begin{pmatrix} [[\nu]]_q & 0 & 0 & 0 \\ 0 & 1 & 0 & -q^{-\nu} \\ 0 & 0 & [[\nu]]_q & 0 \\ 0 & -q^{-\nu} & 0 & 1 \end{pmatrix} \tag{18b}$$

$[[\nu]]_q = 1 - q^{-2\nu}$, $[[\nu]]_q! = [[\nu]]_q[[\nu-1]]_q \dots [[1]]_q$ for $\nu \in \mathbb{N}^+$ and $[[0]]_q! = 1$. As long as $q > 1$, *the overlap matrix is, therefore, positive definite and a* Fock-space orthonormal basis is given by

$$\begin{array}{lll} |0\rangle & |(\nu)2\rangle = ([[\nu]]_q!)^{-1/2}|(23)^\nu 2\rangle & |(\nu)3\rangle = ([[\nu]]_q!)^{-1/2}|(32)^\nu 3\rangle \\ \multicolumn{3}{l}{|(\nu+1)\pm\rangle = (2[[\nu]]_q!(1 \mp q^{-\nu-1}))^{-1/2}[|(23)^{\nu+1}\rangle \pm |(32)^{\nu+1}\rangle] \qquad \nu = 0, 1, 2, \dots} \end{array} \tag{19}$$

together with similar states obtained by adding labels 1, 4, or 14 to (19), i.e. $|1\rangle$, $|1(\nu)2\rangle$, $|1(\nu)3\rangle$, $|1(\nu+1)\pm\rangle$, $|4\rangle$, $|(\nu)24\rangle$, $|(\nu)34\rangle$, $|(\nu+1)4\pm\rangle$, $|14\rangle$, $|1(\nu)24\rangle$, $|1(\nu)34\rangle$ and $|1(\nu+1)4\pm\rangle$. Such a basis contains both symmetric and antisymmetric states under exchange of labels 2 and 3, each of the latter being repeated any number of times.

In the $q \to 1$ limit, however, the states with more than one label 2 or one label 3, as well as those symmetric under exchange of 2 with 3, have a vanishing norm so that the only surviving states are

$$
\begin{gathered}
|0\rangle \qquad |1\rangle \qquad |2\rangle \qquad |3\rangle \qquad |4\rangle \qquad |12\rangle \qquad |13\rangle \qquad |14\rangle \\
|(1)-\rangle = \tfrac{1}{2}[|23\rangle - |32\rangle] \qquad |24\rangle \qquad |34\rangle \qquad |1(1)-\rangle = \tfrac{1}{2}[|123\rangle - |132\rangle] \\
|124\rangle \qquad |134\rangle \qquad |(1)4-\rangle = \tfrac{1}{2}[|234\rangle - |324\rangle] \qquad |1(1)4-\rangle = \tfrac{1}{2}[|1234\rangle - |1324\rangle].
\end{gathered}
\tag{20}
$$

These states can be put into one-to-one correspondence with fermion states with which they can be identified.

In conclusion, we have given two examples of covariant deformed oscillator algebras with a non-standard behaviour. The first example provides enough rules to reorder any pair of operators, while, in the second, one rule is missing. In both cases, however, for $q > 1$, the Fock space contains some states that disappear in the $q \to 1$ limit. A drastic change of statistics is therefore observed as in the case of the quon algebra.

Covariant deformed oscillator algebras with such properties are very common and actually occur for any braid matrix with more than two distinct eigenvalues. The first algebra considered in the present paper, for instance, can be generalized to any quantum group $O_q(2n+1, \mathbb{R})$ while the second algebra can be extended to any $O_q(2n, \mathbb{R})$ or to any $SU_q(n) \times SU_q(m)$.

It is also worth noting that some algebras closer to the quon algebra, in the sense that they contain less relations among creation (or annihilation) operators, can be constructed. Using the projector onto the eigenspace of the $O_q(3, \mathbb{R})$ braid matrix corresponding to $L = 0$ would lead, for instance, to an algebra with a single relation among creation operators.

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